From Entropy Inequalities to Sumset Inequalities

Aref Namayandeh

Department of Information Engineering
The Chinese University of Hong Kong

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Joint work with Chandra Nair and Zhaobang Zhu [NNZ26]
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Outline

1. Background
   - Additive Combinatorics
   - Entropy Basics
2. The Translation Programme
3. Our Result and Proof
   - Main Inequality
   - Proof of Main Inequality
4. Consequences for Sumsets
   - Triangle Inequalities
   - Plünnecke–Ruzsa Inequalities
   - Non-Abelian Extension
5. Conclusion

Big Picture: Additive Combinatorics

Additive combinatorics is concerned with how sumsets

$$A + B := \{a + b : a \in A,\ b \in B\}$$

behave as a function of the underlying sets A and B.

Sets may have a large or small amount of "additive structure".

Notation: $(\mathbb{G}, +)$ denotes an abelian group.

Big Picture: Additive Combinatorics

Many central questions concern growth under addition.

- Doubling constant [Ruz09]: how much does a set expand under addition?

$$K := \frac{|A+A|}{|A|}.$$

- Marton's conjecture / Polynomial Freiman–Ruzsa theorem [GGMT25]: for $A \subseteq \mathbb{F}_2^n$, small additive growth

$$|A+A| \leq K|A|$$

  implies that A is covered by $K^{O(1)}$ translates of a subgroup H with $|H| \leq |A|$.

- Sum–product phenomenon [ES83]: addition and multiplication cannot both have very small growth. For finite nonempty $A \subseteq \mathbb{Z}$,

$$\max\{|A+A|,\ |A \cdot A|\} \geq c|A|^{1+\varepsilon},$$

  for some absolute constants $c, \varepsilon > 0$.

Big Picture: Information Theory

- Shannon entropy $H(X)$: the amount of uncertainty in a random variable X, measured in bits.
- How many bits are needed to describe a source without losing information?
- Mutual information $I(X;Y)$: the amount of information shared by X and Y.
- How many bits per channel use can we send through a noisy channel $p_{Y|X}$?

The Bridge: Additive Combinatorics and Information Theory

For a random variable X supported on a finite nonempty set A,

$$H(X) \leq \log |A|,$$

with equality if and only if $X \sim \mathrm{Unif}(A)$.

The translation programme

Sumset inequalities $\longleftrightarrow$ Entropic inequalities

- When are these statements equivalent?
- Can entropy give direct proofs of sumset inequalities?

Shannon Entropy

For a finite-valued random variable $X \sim p_X$,

$$H(X) := -\sum_x p_X(x) \log p_X(x).$$

- Measures the uncertainty in X (in bits). Equivalently, the information revealed by observing X.
- $H(X) \leq \log |\operatorname{supp}(X)|$, with equality iff X is uniform.

Shannon Entropy

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- Measures the uncertainty in X (in bits). Equivalently, the information revealed by observing X.
- $H(X) \leq \log |\operatorname{supp}(X)|$, with equality iff X is uniform.
- Deterministic maps:
$$H(f(X)) \leq H(X).$$
- Subadditivity:
$$H(X, Y) \leq H(X) + H(Y).$$
- Submodularity: for $X_A := (X_i)_{i \in A}$,
$$H(X_{A \cup B}) + H(X_{A \cap B}) \leq H(X_A) + H(X_B).$$

Conditional Entropy

Conditional entropy:

$$H(Y \mid X) := \sum_x p_X(x)\, H(Y \mid X = x).$$

- The average amount of uncertainty remaining in Y after observing X.
- $0 \le H(Y \mid X) \le H(Y)$. Moreover, $H(Y \mid X) = 0$ if and only if X determines Y: $Y = f(X)$.
- Chain rule:

$$H(X, Y) = H(X) + H(Y \mid X).$$

Mutual Information

Mutual information:

$$\begin{aligned} I(X;Y) &:= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X \mid Y) \\ &= H(Y) - H(Y \mid X). \end{aligned}$$

- The amount of information shared by X and Y.
- $I(X;Y) \geq 0$, and $I(X;Y) = 0$ if and only if $X \perp Y$.

Mutual Information

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Conditional mutual information:

$$\begin{aligned} I(X;Y \mid Z) &:= \sum_z p_Z(z) I(X;Y|Z=z) \\ &= H(X,Z) + H(Y,Z) - H(Z) - H(X,Y,Z). \end{aligned}$$

- The mutual information between X and Y after observing Z.
- $I(X;Y \mid Z) \geq 0$, and $I(X;Y \mid Z) = 0$ if and only if $X \perp Y \mid Z$. Equivalently, we have a Markov chain
$$X \to Z \to Y.$$

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Sumsets and Entropic Inequalities: An Example

Let A, B be finite nonempty subsets of an abelian group $(\mathbb{G}, +)$. If $\operatorname{supp} X \subseteq A$ and $\operatorname{supp} Y \subseteq B$, then

$$H(X+Y) \leq H(X,Y) \leq H(X) + H(Y).$$

Choose a coupling for which $X+Y$ is uniform on $A+B$. Then

$$\log|A+B| \leq \log|A| + \log|B|,$$

so

$$|A+B| \leq |A|\,|B|.$$

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If $X \perp Y$, then

$$H(X+Y) \geq H(X+Y \mid Y) = H(X \mid Y) = H(X).$$

Taking $X \sim \operatorname{Unif}(A)$ gives

$$|A| \leq |A+B|.$$

Sumsets and Entropic Inequalities: Analogues

Several researchers have observed that inequalities in additive combinatorics have analogous entropic inequalities.

Ruzsa's Sum–Difference Inequality [Ruz09]

Let A, B be finite nonempty subsets of an abelian group $(\mathbb{G}, +)$. Then

$$|A|\,|B|\,|A+B| \leq |A-B|^3.$$

An Entropic Analogue: Tao (2010) [Tao10]

Let X, Y be *independent* random variables with finite support on an abelian group $(\mathbb{G}, +)$. Then

$$H(X) + H(Y) + H(X+Y) \leq 3H(X-Y).$$

No direct connection between them.

Ruzsa's Translation Programme

Question

When are sumset inequalities and entropic inequalities formally equivalent?

- Ruzsa 2009 [Ruz09], provided an equivalence theorem for inequalities involving.
- Later, Lau and Nair [LN25] extended the theory to a family of ordinary sumset inequalities.
- The entropy counterpart uses the max over the couplings:

$$\Pi(p_X, p_Y) := \{q_{XY} : q_X = p_X,\ q_Y = p_Y\}.$$

The Equivalence Theorem: Setup

Let $\mathbb{T}$ be a *finitely generated torsion-free* abelian group.
Integer-linear maps:

$$f_1, \dots, f_k, \quad g_1, \dots, g_\ell : \mathbb{T}^n \longrightarrow \mathbb{T}.$$

Positive weights:

$$\alpha_1, \dots, \alpha_k, \quad \beta_1, \dots, \beta_\ell > 0.$$

Let S_i and T_j be the sets of coordinates with nonzero coefficients in f_i and g_j. Assume

$$S_i \cap S_{i'} = \emptyset \qquad \text{for } i \neq i'.$$

Notation:

$$A_S := \prod_{r \in S} A_r, \qquad X_S := (X_r)_{r \in S}.$$

For given individual marginals, write

$$\Pi_S := \Pi\big((p_{X_r})_{r \in S}\big).$$

The Lau–Nair Equivalence Theorem

Under the preceding assumptions, the following are equivalent [LN25, Theorem 1].

Sumset Inequality

For all finite nonempty $A_1, \dots, A_n \subseteq \mathbb{T}$,

$$\prod_{i=1}^{k} |f_i(A_{S_i})|^{\alpha_i} \leq \prod_{j=1}^{\ell} |g_j(A_{T_j})|^{\beta_j}.$$

Entropic Inequality

For all finite-support marginals $p_{X_1}, \dots, p_{X_n}$ on $\mathbb{T}$,

$$\sum_{i=1}^{k} \alpha_i \max_{\Pi_{S_i}} H\big(f_i(X_{S_i})\big) \leq \sum_{j=1}^{\ell} \beta_j \max_{\Pi_{T_j}} H\big(g_j(X_{T_j})\big).$$

Translation Programme Results

For finitely generated torsion-free abelian groups:

- Ruzsa's sumset triangle inequality [LN25]:

$$|A|\,|B+C| \le |A+B|\,|A+C|.$$

$$H(X) + \max_{\Pi(p_Y,p_Z)} H(Y+Z) \le \max_{\Pi(p_X,p_Y)} H(X+Y) + \max_{\Pi(p_X,p_Z)} H(X+Z).$$

- Ruzsa's sum–difference inequality [LN25]:

$$|A|\,|B|\,|A+B| \le |A-B|^3.$$

$$H(X) + H(Y) + \max_{\Pi(p_X,p_Y)} H(X+Y) \le 3 \max_{\Pi(p_X,p_Y)} H(X-Y).$$

- Fractional-partition [GMR10, LNZ26]: For $\gamma(S) \ge 0$ with $\sum_{S \ni i} \gamma(S) = 1$ for every $i \in [n]$,

$$|A_1 + \cdots + A_n| \le \prod_{\emptyset \ne S \subseteq [n]} \left| \sum_{i \in S} A_i \right|^{\gamma(S)}.$$

$$H\left(\sum_{i=1}^{n} X_i\right) \le \sum_{\emptyset \ne S \subseteq [n]} \gamma(S) \max_{\Pi_S} H\left(\sum_{i \in S} X_i\right).$$

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This Work

Conjecture: Lau–Nair–Zhu [LNZ26]

For finite-support marginals on any abelian group,

$$H(X) + \max_{\Pi(p_Y, p_Z)} H(Y+Z) \leq \max_{\Pi(p_X, p_Y)} H(X+Y) + \max_{\Pi(p_X, p_Z)} H(X+Z).$$

The open problem was to find a purely information-theoretic proof.

This Work

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Question 1

Can we remove the finitely generated torsion-free assumption for this and other inequalities?

Yes, The equivalence theorem can be extended to hold on arbitrary abelian groups.

This Work

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Question 1

Can we remove the finitely generated torsion-free assumption for this and other inequalities?

Yes, The equivalence theorem can be extended to hold on arbitrary abelian groups.

Question 2

Can we find a stand-alone information-theoretic proof?

Yes, by maximizing entropy over a fibre. We prove a stronger inequality and derive further entropic and sumset inequalities.

Our Main Inequality

We prove this inequality:

Theorem ([NNZ26])

Let p be any joint distribution of finite-support random variables X, Y, Z on an abelian group, with marginals p_X, p_Y, p_Z. Then

$$H(X) + H_p(X+Y+Z) \leq \max_{\Pi(p_X, p_Y)} H(X+Y) + \max_{\Pi(p_X, p_Z)} H(X+Z).$$

Proof: Fixing the Fibre

Write $U = X + Y + Z$ and define

$$\mathcal{M}(p_X, p_Y, p_Z, p_U) := \{q_{XYZ} : q_X = p_X,\ q_Y = p_Y,\ q_Z = p_Z,\ q_U = p_U\}.$$

We claim that there exists $q \in \mathcal{M}$ such that

$$H(X) + H(U) \leq H_q(X + Y) + H_q(X + Z).$$

Then

$$\begin{aligned} H(X) + H_p(X + Y + Z) &\leq \max_{q \in \mathcal{M}} \big[H_q(X + Y) + H_q(X + Z)\big] \\ &\leq \max_{q \in \Pi(p_X, p_Y, p_Z)} \big[H_q(X + Y) + H_q(X + Z)\big] \\ &\leq \max_{\Pi(p_X, p_Y)} H(X + Y) + \max_{\Pi(p_X, p_Z)} H(X + Z). \end{aligned}$$

Proof: Choosing the Maximum-Entropy Coupling

The feasible set $\mathcal{M}$ is nonempty, convex, and compact, and $H(X)$ is continuous and strictly concave in p_X.
Choose

$$q^\star \in \arg\max_{q \in \mathcal{M}} H_q(X, Y, Z).$$

Claim

Under $q^\star$,

$$X \to \overbrace{X+Y}^{R} \to \overbrace{X+Y+Z}^{U},$$

and

$$X \to \overbrace{X+Z}^{S} \to \overbrace{X+Y+Z}^{U}.$$

Let $R = X + Y$, $S = X + Z$, and $U = X + Y + Z$.

Proof of the Claim: Recombination

The transformation $(X, Y, Z) \longleftrightarrow (X, R, U)$ is bijective. Consider $q^{\star}_{XRU}$. Define a distribution on (X, R, U) by

$$\widetilde{q}_{XRU}(x, r, u) := q^{\star}_{R}(r)\, q^{\star}_{X|R}(x \mid r)\, q^{\star}_{U|R}(u \mid r).$$

$$X \to R \to U$$

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$$X \to R \to U$$

This preserves

$$\widetilde{q}_{XR} = q^\star_{XR}, \qquad \widetilde{q}_{RU} = q^\star_{RU}.$$

The induced distribution $\widetilde{q}$ preserves the marginals of X, Y, Z, U. Therefore,

$$\widetilde{q} \in \mathcal{M}.$$

Intuition

The recombination increases the entropy by $I(X; U \mid R)$, contradicting the maximality of $q^\star$ unless $I(X; U \mid R) = 0$.

Proof of the Claim: Maximality Forces Conditional Independence

The transformation $(X, Y, Z) \longleftrightarrow (X, R, U)$ is bijective. Therefore,

$$\begin{aligned}
\underbrace{H_{\widetilde{q}}(X, R, U)}_{H_{\widetilde{q}}(X,Y,Z)} &= H_{\widetilde{q}}(R) + H_{\widetilde{q}}(X \mid R) + H_{\widetilde{q}}(U \mid R) \\
&= H_{q^\star}(R) + H_{q^\star}(X \mid R) + H_{q^\star}(U \mid R) \\
&= \underbrace{H_{q^\star}(X, R, U)}_{H_{q^\star}(X,Y,Z)} + I_{q^\star}(X; U \mid R).
\end{aligned}$$

Proof of the Claim: Maximality Forces Conditional Independence

The transformation $(X, Y, Z) \longleftrightarrow (X, R, U)$ is bijective. Therefore,

$$\begin{aligned}\underbrace{H_{\tilde{q}}(X, R, U)}_{H_{\tilde{q}}(X,Y,Z)} &= H_{\tilde{q}}(R) + H_{\tilde{q}}(X \mid R) + H_{\tilde{q}}(U \mid R) \\ &= H_{q^\star}(R) + H_{q^\star}(X \mid R) + H_{q^\star}(U \mid R) \\ &= \underbrace{H_{q^\star}(X, R, U)}_{H_{q^\star}(X,Y,Z)} + I_{q^\star}(X; U \mid R).\end{aligned}$$

By maximality of $q^\star$ and nonnegativity of conditional mutual information,

$$\boxed{I_{q^\star}(X; U \mid R) = 0.}$$

Similarly, recombining through $S = X + Z$, gives

$$\boxed{I_{q^\star}(X; U \mid S) = 0.}$$

Proof: The Entropy Identity

For any coupling, the chain rule gives

$$\begin{aligned} I(R;S \mid X) &= H(X,R) + H(X,S) - H(X) - H(X,R,S), \\ I(R;S \mid U) &= H(U,R) + H(U,S) - H(U) - H(R,S,U), \\ I(X;U \mid R) &= H(X,R) + H(U,R) - H(R) - H(X,R,U), \\ I(X;U \mid S) &= H(X,S) + H(U,S) - H(S) - H(X,S,U). \end{aligned}$$

The four red terms are equal:

$$R + S = X + U.$$

Therefore,

$$\begin{aligned} & H(R) + H(S) - H(X) - H(U) && (1) \\ & \quad = I(R;S \mid X) + I(R;S \mid U) - I(X;U \mid R) - I(X;U \mid S). && (2) \end{aligned}$$

Proof: Finishing with the Entropy Identity

Under $q^\star$, the two negative terms vanish:

$$\begin{aligned} & H_{q^\star}(R) + H_{q^\star}(S) - H(X) - H(U) \\ & \quad = I_{q^\star}(R; S \mid X) + I_{q^\star}(R; S \mid U) \geq 0. \end{aligned}$$

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Therefore,

$$\boxed{H(X) + H_p(X + Y + Z) \leq H_{q^\star}(X + Y) + H_{q^\star}(X + Z).}$$

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The Coupling Triangle Inequality

$$H(X) + H_p(X+Y+Z) \leq \max_{\Pi(p_X, p_Y)} H(X+Y) + \max_{\Pi(p_X, p_Z)} H(X+Z).$$

Choose $X \perp (Y, Z)$. Then

$$H(X+Y+Z) \geq H(X+Y+Z \mid X) = H(Y+Z).$$

Applying our theorem and maximizing over $\Pi(p_Y, p_Z)$ gives

$$\boxed{H(X) + \max_{\Pi(p_Y, p_Z)} H(Y+Z) \leq \max_{\Pi(p_X, p_Y)} H(X+Y) + \max_{\Pi(p_X, p_Z)} H(X+Z).}$$

Ruzsa's Sumset Triangle Inequality

Choose

$$X \sim \mathrm{Unif}(A), \qquad Y + Z \sim \mathrm{Unif}(B + C),$$

where $\operatorname{supp} Y \subseteq B$ and $\operatorname{supp} Z \subseteq C$.
Then

$$\begin{aligned} \log|A| + \log|B + C| &= H(X) + \max_{\Pi(p_Y, p_Z)} H(Y + Z) \\ &\leq \max_{\Pi(p_X, p_Y)} H(X + Y) + \max_{\Pi(p_X, p_Z)} H(X + Z) \\ &\leq \log|A + B| + \log|A + C|. \end{aligned}$$

Therefore, we recover Ruzsa's sumset triangle inequality:

$$\boxed{|A|\,|B + C| \leq |A + B|\,|A + C|.}$$

Magnification Ratios

For finite nonempty $A, D \subseteq \mathbb{G}$, define

$$\mu(A, D) := \min_{\emptyset \neq S \subseteq A} \frac{|S + D|}{|S|}.$$

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The Plünnecke–Ruzsa inequalities [Ruz09]:

$$\mu(A, B_1 + \cdots + B_k) \leq \prod_{i=1}^{k} \frac{|A + B_i|}{|A|}.$$

In particular [Plü70],

$$\mu(A, kB) \leq \left(\frac{|A + B|}{|A|}\right)^k.$$

Entropic Characterization of the Magnification Ratio

For a fixed distribution p_X, define

$$\Phi_D(p_X) := \max_{\substack{p_{Y|X} \\ \operatorname{supp} Y \subseteq D}} [H(X+Y) - H(X)].$$

Theorem (Lau–Nair [LN25])

$$\log \mu(A, D) = \min_{\operatorname{supp} p_X \subseteq A} \Phi_D(p_X).$$

Entropic Plünnecke–Ruzsa Inequalities

For any joint distribution p_{XW} with $W \in B + C$, choose

$$W = Y + Z, \qquad Y \in B, \quad Z \in C.$$

Our inequality gives

$$\begin{aligned} H(X+W) - H(X) &\le \max_{\Pi(p_X, p_Y)} H(X+Y) - H(X) \\ &\quad + \max_{\Pi(p_X, p_Z)} H(X+Z) - H(X) \\ &\le \Phi_B(p_X) + \Phi_C(p_X). \end{aligned}$$

Maximizing over $p_{W|X}$ with $W \in B + C$ gives

$$\boxed{\Phi_{B+C}(p_X) \le \Phi_B(p_X) + \Phi_C(p_X).}$$

Entropic Plünnecke–Ruzsa Inequalities

Using induction, we obtain an entropic counterpart of the Plünnecke–Ruzsa inequalities [NNZ26]:

$$\Phi_{B_1+\cdots+B_k}(p_X) \le \sum_{i=1}^{k} \Phi_{B_i}(p_X),$$

and consequently,

$$\boxed{\min_{\operatorname{supp} p_X \subseteq A} \Phi_{B_1+\cdots+B_k}(p_X) \le \min_{\operatorname{supp} p_X \subseteq A} \sum_{i=1}^{k} \Phi_{B_i}(p_X).}$$

Plünnecke–Ruzsa Inequalities

Write $D = B_1 + \cdots + B_k$. Using the identity $\log \mu(A, D) = \min_{\operatorname{supp} p_X \subseteq A} \Phi_D(p_X)$, for every nonempty $S \subseteq A$,

$$\begin{aligned}
\log \mu(A, D) &= \min_{\operatorname{supp} p_X \subseteq A} \Phi_D(p_X) \\
&\leq \min_{\operatorname{supp} p_X \subseteq A} \sum_{i=1}^{k} \Phi_{B_i}(p_X) \\
&\leq \sum_{i=1}^{k} \Phi_{B_i}(\mathrm{Unif}(S)) \\
&\leq \sum_{i=1}^{k} \log \frac{|S + B_i|}{|S|}.
\end{aligned}$$

Therefore [NNZ26],

$$\boxed{\mu(A, D) \leq \min_{\emptyset \neq S \subseteq A} \prod_{i=1}^{k} \frac{|S + B_i|}{|S|} \leq \prod_{i=1}^{k} \frac{|A + B_i|}{|A|}.}$$

Non-Abelian Extension

Using the same method, but with

$$U = Y \circ X \circ Z, \quad R = Y \circ X, \quad S = X \circ Z,$$

we prove [NNZ26]

$$\boxed{H(X) + H(Y \circ X \circ Z) \leq \max_{\Pi(p_X, p_Y)} H(Y \circ X) + \max_{\Pi(p_X, p_Z)} H(X \circ Z).}$$

This implies a non-abelian sumset inequality:

$$\boxed{\sum_{x \in A} |B \circ x \circ C| \leq |B \circ A|\, |A \circ C|.}$$

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Summary

We give a direct information-theoretic proof of max-coupling inequalities, without independence or torsion-free assumptions, and without typical-set argument or translation programme.

Combinatorial Consequences

- Ruzsa's sumset triangle inequality.
- The Plünnecke–Ruzsa inequalities.
- A non-abelian Ruzsa sum inequality.

Thank you!

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